LIMIT THEOREMS FOR ADDITIVE C-FREE CONVOLUTION

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Abstract. In this paper we determine the limiting distributional behavior for sums of infinitesimal c-free random variables. We show that the weak convergence of classical convolution and that of c-free convolution are equivalent for measures in an infinitesimal triangular array, where the measures may have unbounded support. Moreover, we use these limit theorems to study the c-free infinite divisibility and stability. These results are obtained by complex analytic methods without reference to the combinatorics of c-free convolution.

1. Introduction

The theory of the conditionally free (abbreviated as c-free) random variables was introduced by Bożejko, Leinert and Speicher in [9], as a generalization of Voiculescu's freeness to the algebras with two states. The concept of c-freeness leads to a binary operation, called additive c-free convolution, on pairs of compactly supported probability measures on the real line. The c-free analogues of central and Poisson limit theorems for identically distributed summands were also proved in [9]. The development of the c-free probability theory relies heavily on the combinatorics of non-crossing partitions. The nature of the combinatorial tools makes it difficult to discuss limit theorems when the measures do not have finite moments. Even for finite moments the limit theorems proved in [9] and [10] require subtle combinatorics arguments.

The aim of this paper is to provide an analytic approach to study the asymptotic distributional behavior of additive c-free convolution. As shown in [17], the same approach also works in the multiplicative context. The extension of (additive) c-free convolution to measures with unbounded support was done by Belinschi [2]. His work provided useful inspirations for some of the analytic questions in our approach, as will be seen below.

The remainder of this paper is organized as follows. In Section 2 we deal with the analytic problems involved in using an analogue of Voiculescu's R-transform for measures without bounded support, and we extend the definition of c-free convolution to pairs of arbitrary measures using this transform. Section 3 contains the main result of this paper (Theorem 3.5), which provides necessary and sufficient conditions for

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the weak convergence of c-free convolution of measures in an infinitesimal array. In Section 4 we present various characterizations of c-free infinite divisibility, which extend the results in [15] for pairs of compactly supported measures. Section 5 contains a brief discussion of c-free stability.

2. Setting and Basic Properties

In this section we focus on the analytic apparatus needed for the calculation of c-free convolution. Most of the results we quoted from the literature were developed for studying the free and boolean convolutions. We refer the reader to the book [20] for a comprehensive introduction to free probability theory, and to the papers [18, 4] for a detailed treatment of boolean probability theory.

2.1. **Cauchy transforms and c-free convolution.** Denote by $\mathcal{M}$ the family of all Borel probability measures on the real line $\mathbb{R}$ and set $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$, $\mathbb{C}^- = -\mathbb{C}^+$. We associate each measure $\mu \in \mathcal{M}$ its *Cauchy transform*

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\, d\mu(t), \qquad z \in \mathbb{C}^+,$$

and its reciprocal $F_\mu = 1/G_\mu : \mathbb{C}^+ \to \mathbb{C}^+$. The measure μ can be recovered from G_μ as the weak*-limit of the measures

$$d\nu_y(x) = -\frac{1}{\pi}\Im G_\mu(x+iy)\, dx$$

as $y \to 0^+$. For $\alpha, \beta > 0$, we define the cone $\Gamma_\alpha = \{x+iy \in \mathbb{C}^+ : |x| < \alpha y\}$ and the truncated cone $\Gamma_{\alpha,\beta} = \{x+iy \in \Gamma_\alpha : y > \beta\}$. As shown in [7], we have $\Im z \leq \Im F_\mu(z)$ for $z \in \mathbb{C}^+$ and

$$F_\mu(z) = z(1+o(1)), \qquad z \in \mathbb{C}^+, \tag{2.1}$$

as $z \to \infty$ *nontangentially* (i.e., $|z| \to \infty$ but z stays within a cone Γ_α for some $\alpha > 0$.) The measure μ is uniquely determined by the function F_μ, and conversely, any analytic function $F : \mathbb{C}^+ \to \mathbb{C}^+$ so that $F(z) = z(1+o(1))$ as $z \to \infty$ nontangentially is of the form F_μ for a unique probability measure μ on $\mathbb{R}$.

Property (2.1) also implies that, for every $\alpha > 0$, there exists $\beta = \beta(\mu,\alpha) > 0$ such that the function F_μ has a left inverse F_μ^{-1} (relative to composition) defined in $\Gamma_{\alpha,\beta}$. Moreover, we see that $F_\mu^{-1}(z) = z(1+o(1))$ as $z \to \infty$ nontangentially. For $\mu, \nu \in \mathcal{M}$, the *additive free convolution* $\mu \boxplus \nu \in \mathcal{M}$ is characterized [7] by the identity

$$F_{\mu\boxplus\nu}^{-1}(z) + z = F_\mu^{-1}(z) + F_\nu^{-1}(z),$$

where z is in a truncated cone $\Gamma_{\alpha,\beta}$ contained in the domain of all involved functions.

For a measure $\mu \in \mathcal{M}$, observe that the function $E_\mu(z) = z - F_\mu(z)$ takes values in $\mathbb{C}^- \cup \mathbb{R}$ and $E_\mu(z) = o(|z|)$ as $z \to \infty$ nontangentially. Conversely, any analytic function $E : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ with these properties is of the form E_μ for a unique

probability measure μ. The *additive boolean convolution* $\mu \uplus \nu \in \mathcal{M}$ of two measures $\mu, \nu \in \mathcal{M}$ is characterized [18, 4] by

$$E_{\mu \uplus \nu}(z) = E_\mu(z) + E_\nu(z), \qquad z \in \mathbb{C}^+.$$

The theory of c-free convolution for pairs of compactly supported probability measures was first studied in [9]. The c-free convolution $(\mu_1, \nu_1) \boxplus_{\mathrm{c}} (\mu_2, \nu_2)$ of such pairs is again a pair of compactly supported probability measures $(\widetilde{\mu}, \widetilde{\nu})$, where the measure $\widetilde{\nu} = \nu_1 \boxplus \nu_2$. In order to describe the measure $\widetilde{\mu}$, these authors further introduced, for a pair of compactly supported measures (μ, ν), the analytic function

$$C_{(\mu,\nu)}(z) = z \left[E_\mu \left(G_\nu^{-1}(z) \right) \right],$$

where the inversion of G_ν is carried out in a neighborhood of ∞, and they proved that

$$C_{(\widetilde{\mu},\widetilde{\nu})}(z) = C_{(\mu_1,\nu_1)}(z) + C_{(\mu_2,\nu_2)}(z).$$

The starting point for the treatment of measures with unbounded support is observing that, for arbitrary measures $\mu, \nu \in \mathcal{M}$, the function $C_{(\mu,\nu)}$ is actually defined in an appropriate domain. For measures $\mu, \nu \in \mathcal{M}$, we introduce a new function

$$\Phi_{(\mu,\nu)}(z) = E_\mu \left(F_\nu^{-1}(z) \right) \tag{2.2}$$

in a truncated cone $\Gamma_{\alpha,\beta}$ where the function F_ν^{-1} is defined. The function $\Phi_{(\mu,\nu)}$ is obtained from the function $C_{(\mu,\nu)}(z)/z$ by a change of variable $z \mapsto 1/z$, and is more suitable for our purposes. It is easy to verify that we have

$$\Phi_{(\widetilde{\mu},\widetilde{\nu})}(z) = \Phi_{(\mu_1,\nu_1)}(z) + \Phi_{(\mu_2,\nu_2)}(z)$$

in the case of compactly supported measures.

We will require the following result from [5], whose proof is based on the Cauchy integral formula.

Lemma 2.1. *Let $\alpha, \beta, \varepsilon$ be positive numbers, and let $\phi : \Gamma_{\alpha,\beta} \to \mathbb{C}$ be an analytic function such that $|\phi(z)| \leq \varepsilon |z|$ for every $z \in \Gamma_{\alpha,\beta}$. Then, for every $\alpha' < \alpha$ and $\beta' > \beta$, there exists $K > 0$ such that the derivative $\phi'(z)$ is estimated as follows*

$$|\phi'(z)| \leq K\varepsilon, \qquad z \in \Gamma_{\alpha',\beta'}.$$

The following result was first noted in [2].

Proposition 2.2. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$, and let $\nu = \nu_1 \boxplus \nu_2$. Suppose that both $F_{\nu_1}^{-1}$ and $F_{\nu_2}^{-1}$ are defined in a cone $\Gamma_{\alpha,\beta}$. Then there exists another truncated cone $\Gamma_{\alpha',\beta'} \subset \Gamma_{\alpha,\beta}$ such that the function*

$$\Phi(z) = \Phi_{(\mu_1,\nu_1)}(z) + \Phi_{(\mu_2,\nu_2)}(z), \qquad z \in \Gamma_{\alpha',\beta'},$$

is of the form $\Phi_{(\mu,\nu)}$ for a unique probability measure μ on $\mathbb{R}$.

Proof. Note that (2.1) shows that $F_\nu(z) \in \Gamma_{\alpha,\beta}$ as $z \to \infty$ nontangentially. To prove the proposition, it suffices to show that the function $E(z) = \Phi(F_\nu(z))$ is of the form $E_\mu(z)$ for a unique probability measure $\mu \in \mathcal{M}$, that is, to show that the function $E(z)$ extends analytically to $\mathbb{C}^+$ and $E(z)/z \to 0$ as $z \to \infty$ nontangentially.

To this purpose, we appeal to a subordination result in [3] (see also [11]) for free convolution $\nu_1 \boxplus \nu_2$, namely, there exist unique analytic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that $\omega_j(z) = z(1 + o(1))$, $j = 1, 2$, as $z \to \infty$ nontangentially and $F_\nu(z) = F_{\nu_1}(\omega_1(z)) = F_{\nu_2}(\omega_2(z))$ for all $z \in \mathbb{C}^+$. Then, by (2.2), we have

$$E(z) = E_{\mu_1}(\omega_1(z)) + E_{\mu_2}(\omega_2(z))$$

in an open subset of $\mathbb{C}^+$, and hence the function $E(z)$ extends analytically to the entire upper half-pane $\mathbb{C}^+$.

On the other hand, Lemma 2.1 shows that the derivatives $E'_{\mu_j}(z) = o(1)$, $j = 1, 2$, as $z \to \infty$ nontangentially. It follows that there exists $M > \beta$ such that

$$|E(z) - E_{\mu_1}(z) - E_{\mu_2}(z)| \leq |\omega_1(z) - z| + |\omega_2(z) - z|,$$

for $z \in \Gamma_{\alpha,M}$, and hence we conclude that $E(z)/z \to 0$ as $z \to \infty$ nontangentially. Thus the proof is complete. $\square$

Proposition 2.2 allows us to make the following definition which will be used throughout the rest of this paper.

Definition 2.3. Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$, and let $\nu = \nu_1 \boxplus \nu_2$. The additive c-free convolution $(\mu_1, \nu_2) \boxplus_{\mathrm{c}} (\mu_2, \nu_2)$ is the pair (μ, ν), where μ is the unique probability measure provided by Proposition 2.2.

We will also use the somewhat abused notation

$$\mu = \mu_1 \boxplus_{\mathrm{c}} \mu_2.$$

Indeed, $\mu_1 \boxplus_{\mathrm{c}} \mu_2$ depends on ν_1 and ν_2 as well. We choose this shorter notation because the asymptotic behavior of free convolution $\boxplus$ is well understood (see [13], and [8] for a different approach), and we would like to address convergence issues on the first component of c-free convolution. Our second remark is that the operation $\boxplus_{\mathrm{c}}$ is commutative and associative by Proposition 2.2, and it reduces to the original c-free convolution introduced in [9] in the case of compactly supported measures.

2.2. **Weak convergence of probability measures.** If μ_n and μ are elements of $\mathcal{M}$, or more generally, finite Borel measures on $\mathbb{R}$, we say that μ_n converges *weakly* to μ if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, d\mu_n(t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, d\mu(t)$$

for every bounded continuous function f on $\mathbb{R}$. The weak convergence of measures requires tightness. Recall that a family $\mathcal{F}$ of finite Borel measures on $\mathbb{R}$ is *tight* if

$$\lim_{y\to+\infty}\sup_{\mu\in\mathcal{F}}\mu(\{t:\,|t|>y\})=0.$$

Any tight sequence of probability measures has a subsequence which converges weakly to a probability measure.

We note for further reference that weak convergence of probability measures can be translated in terms of convergence properties of the corresponding functions E and Φ.

Proposition 2.4. *Let $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ be two sequences in $\mathcal{M}$.*

(1) *The sequence μ_n converges weakly to a measure $\mu\in\mathcal{M}$ if and only if there exists a truncated cone Γ such that the sequence E_{μ_n} converges uniformly on the compact subsets of Γ to a function E, and $E_{\mu_n}(z)=o(|z|)$ uniformly in n as $|z|\to\infty$, $z\in\Gamma$. Moreover, we have $E=E_\mu$ in this situation.*

(2) *Assume that the sequence ν_n converges weakly to a measure $\nu\in\mathcal{M}$. Then the sequence μ_n converges weakly to a measure $\mu\in\mathcal{M}$ if and only if there exist $\alpha,\beta>0$ such that the functions $\Phi_{(\mu_n,\nu_n)}$ are defined in the cone $\Gamma_{\alpha,\beta}$ for every n, $\lim_{n\to\infty}\Phi_{(\mu_n,\nu_n)}(iy)$ exists for every $y>\beta$ and $\Phi_{(\mu_n,\nu_n)}(iy)=o(y)$ uniformly in n as $y\to\infty$. Moreover, in this case we have $\lim_{n\to\infty}\Phi_{(\mu_n,\nu_n)}(iy)=\Phi_{(\mu,\nu)}(iy)$ for every $y>\beta$.*

Proof. We refer to [5] for the proof of (1). To prove (2), note first that the existence of the truncated cone $\Gamma_{\alpha,\beta}$ is provided by the weak convergence of the sequence $\{\nu_n\}_{n=1}^{\infty}$ (see [5, Proposition 2.3]). Moreover, the sequence $F_{\nu_n}^{-1}$ converges uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$ to the function F_ν^{-1}, and $F_{\nu_n}^{-1}(z)=z(1+o(1))$ uniformly in n as $z\to\infty$, $z\in\Gamma_{\alpha,\beta}$.

Assume that the measures μ_n converge weakly to a measure μ. Then (1) and Lemma 2.1 imply that the derivatives $E'_\mu(z)=o(1)$ and $E'_{\mu_n}(z)=o(1)$ uniformly in n as $z\to\infty$ nontangentially. It follows that there exists $M>\beta$ such that

$$\begin{aligned}
\left|\Phi_{(\mu_n,\nu_n)}(z)-\Phi_{(\mu,\nu)}(z)\right| &= \left|E_{\mu_n}\left(F_{\nu_n}^{-1}(z)\right)-E_\mu\left(F_\nu^{-1}(z)\right)\right|\\
&\le \left|E_{\mu_n}\left(F_{\nu_n}^{-1}(z)\right)-E_{\mu_n}\left(F_\nu^{-1}(z)\right)\right|\\
&\quad+\left|E_{\mu_n}\left(F_\nu^{-1}(z)\right)-E_\mu\left(F_\nu^{-1}(z)\right)\right|\\
&\le \left|F_{\nu_n}^{-1}(z)-F_\nu^{-1}(z)\right|+\left|E_{\mu_n}\left(F_\nu^{-1}(z)\right)-E_\mu\left(F_\nu^{-1}(z)\right)\right|
\end{aligned}$$

for every $n\in\mathbb{N}$ and $z\in\Gamma_{\alpha,M}$. Hence (1) implies that $\Phi_{(\mu_n,\nu_n)}(z)=o(|z|)$ uniformly in n as $z\to\infty$, $z\in\Gamma_{\alpha,\beta}$. The family $\{\Phi_{(\mu_n,\nu_n)}\}_{n=1}^{\infty}$ is normal, and hence it has subsequences which converge uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$. Moreover, the above estimate and (1) actually imply that the limit of such a subsequence must be

the function $\Phi_{(\mu,\nu)}$. Therefore we conclude that the entire sequence $\{\Phi_{(\mu_n,\nu_n)}\}_{n=1}^{\infty}$ converges uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$ to the function $\Phi_{(\mu,\nu)}$. In particular, these results hold for $z = iy$, $y > \beta$.

Conversely, let us assume that $\lim_{n\to\infty} \Phi_{(\mu_n,\nu_n)}(iy)$ exists for every $y > \beta$ and $\Phi_{(\mu_n,\nu_n)}(iy) = o(y)$ uniformly in n as $y \to \infty$. We first show that the sequence $\{\mu_n\}_{n=1}^{\infty}$ is tight. Let us define $u_n = u_n(y) = F_{\nu_n}^{-1}(iy) = iy + \phi_{\nu_n}(iy)$ for $y > \beta$, and also observe that $\phi_{\nu_n}(iy) = o(y)$ uniformly in n as $y \to \infty$ by the assumption on the weak convergence of $\{\nu_n\}_{n=1}^{\infty}$. Then we have

$$u_n - F_{\mu_n}(u_n) = E_{\mu_n}(u_n) = \Phi_{(\mu_n,\nu_n)}(iy) = o(y)$$

uniformly in n as $y \to \infty$. Moreover, note that

$$|G_{\mu_n}(u_n(y))| \le \frac{1}{\Im u_n} = \frac{1}{y + o(y)}$$

uniformly in n as $y \to \infty$. Hence, we conclude that $u_n^2 G_{\mu_n}(u_n) - u_n = o(y)$ uniformly in n as $y \to \infty$. On the other hand, since $u_n = iy + o(y)$ uniformly in n as $y \to \infty$, there exists $M > \beta$ such that

$$\frac{t^2}{(\Re u_n(y) - t)^2 + (\Im u_n(y))^2} \ge \frac{1}{8}, \qquad t \in \mathbb{R},\ |t| \ge y > M,$$

for every n. Finally, putting everything together, we have

$$\begin{aligned} -\frac{1}{y}\Im\left(u_n^2 G_{\mu_n}(u_n) - u_n\right) &= \frac{\Im u_n}{y}\int_{-\infty}^{\infty} \frac{t^2}{(\Re u_n - t)^2 + (\Im u_n)^2}\, d\mu_n(t) \\ &\ge \frac{\Im u_n}{y}\int_{|t|\ge y} \frac{1}{8}\, d\mu_n(t) = \frac{\Im u_n}{8y}\, \mu_n(\{t : |t| \ge y\}), \end{aligned}$$

for every n and $y > M$, which implies that $\{\mu_n\}_{n=1}^{\infty}$ is tight. If $\mu \in \mathcal{M}$ is a weak cluster point of $\{\mu_n\}_{n=1}^{\infty}$, then the first part of the proof shows that the function $\Phi_{(\mu,\nu)}$ is uniquely determined and hence so is the measure μ. Therefore the sequence μ_n converges weakly to the measure μ. □

Note that, in case $\nu_n = \delta_0$, Proposition 2.4 gives the equivalence between the weak convergence of $\{\mu_n\}_{n=1}^{\infty}$ and convergence properties of $\{E_{\mu_n}(iy)\}_{n=1}^{\infty}$.

2.3. **Infinite divisibility.** A pair of probability measures (μ, ν) is said to be $\boxplus_c$-*infinitely divisible* if, for every $n \in \mathbb{N}$, there exist measures $\mu_n, \nu_n \in \mathcal{M}$ such that

$$(\mu,\nu) = \underbrace{(\mu_n,\nu_n) \boxplus_{\mathrm{c}} (\mu_n,\nu_n) \boxplus_{\mathrm{c}} \cdots \boxplus_{\mathrm{c}} (\mu_n,\nu_n)}_{n\,\text{times}},$$

in other words, we have

$$\mu = \underbrace{\mu_n \boxplus_{\mathrm{c}} \mu_n \boxplus_{\mathrm{c}} \cdots \boxplus_{\mathrm{c}} \mu_n}_{n\,\text{times}} \qquad \text{and} \qquad \nu = \underbrace{\nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n}_{n\,\text{times}}.$$

The notion of infinite divisibility related to other convolutions is defined analogously.

The Lévy-Hinčin formula (see [12]) characterizes the infinite divisibility relative to classical convolution $*$ of a probability measure in terms of its Fourier transform. Namely, a measure $\nu \in \mathcal{M}$ is $*$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{R}$ such that the Fourier transform $\widehat{\nu}$ of the measure ν is given by

$$\widehat{\nu}(t) = \exp\left[i\gamma t + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{1+x^2}{x^2}\, d\sigma(x)\right], \qquad t \in \mathbb{R}. \tag{2.3}$$

The free analogue of Lévy-Hinčin formula for a $\boxplus$-infinitely divisible probability measure was proved in [19, 7]. A measure $\nu \in \mathcal{M}$ is $\boxplus$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{R}$ such that

$$F_\nu^{-1}(z) = \gamma + z + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma(t), \qquad z \in \mathbb{C}^+. \tag{2.4}$$

In other words, the function F_ν^{-1} can be extended analytically to $\mathbb{C}^+$ if the measure ν is $\boxplus$-infinitely divisible.

Every measure $\nu \in \mathcal{M}$ is $\uplus$-infinitely divisible [18]. The reason for this is that every analytic self-mapping of $\mathbb{C}^+$ has a Nevanlinna integral representation [1]. In particular, the function E_ν can be written as

$$E_\nu(z) = \gamma + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma(t), \qquad z \in \mathbb{C}^+, \tag{2.5}$$

where $\gamma \in \mathbb{R}$ and σ is a finite positive Borel measure on $\mathbb{R}$.

In the sequel, we will use the notations $\nu_*^{\gamma,\sigma}$, $\nu_\boxplus^{\gamma,\sigma}$ and $\nu_\uplus^{\gamma,\sigma}$ to denote respectively the $*$-, $\boxplus$-, and $\uplus$-infinitely divisible measures that are uniquely determined by γ and σ via the formulas (2.3), (2.4) and (2.5).

3. Proof of the Main Result

Let $\{k_n\}_{n=1}^\infty$ be a sequence of positive integers, and let $\{c_n\}_{n=1}^\infty$ and $\{c'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{R}$. Consider two *infinitesimal* triangular arrays $\{\mu_{nk} : n \in \mathbb{N}, 1 \leq k \leq k_n\}$ and $\{\nu_{nk} : n \in \mathbb{N}, 1 \leq k \leq k_n\}$ in $\mathcal{M}$. Here the infinitesimality of the array $\{\mu_{nk}\}_{n,k}$ means that

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \mu_{nk}(\{t \in \mathbb{R} : |t| \geq \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. The goal of this section is to study the asymptotic behavior of the sequence $\{(\mu_n, \nu_n)\}_{n=1}^\infty$, where

$$(\mu_n, \nu_n) = (\delta_{c_n}, \delta_{c'_n}) \boxplus_{\mathrm{c}} (\mu_{n1}, \nu_{n1}) \boxplus_{\mathrm{c}} (\mu_{n2}, \nu_{n2}) \boxplus_{\mathrm{c}} \cdots \boxplus_{\mathrm{c}} (\mu_{nk_n}, \nu_{nk_n}),$$

and δ_c denotes the Dirac point mass at $c \in \mathbb{R}$.

To this purpose, we introduce the measures μ_{nk}° by setting

$$d\mu_{nk}^\circ(t) = d\mu_{nk}(t + a_{nk}),$$

where the numbers $a_{nk} \in [-1,1]$ are given by

$$a_{nk} = \int_{|t|<1} t \, d\mu_{nk}(t). \tag{3.1}$$

Note that the array $\{\mu_{nk}^{\circ}\}_{n,k}$ is infinitesimal and $\lim_{n\to\infty} \max_{1\le k\le k_n} |a_{nk}| = 0$.

We also associate each measure μ_{nk}° an analytic function

$$f_{nk}(z) = \int_{-\infty}^{\infty} \frac{tz}{z-t} \, d\mu_{nk}^{\circ}(t), \qquad z \in \mathbb{C}^+.$$

Observe that $\Im f_{nk}(z) < 0$ for all $z \in \mathbb{C}^+$ unless the measure $\mu_{nk}^{\circ} = \delta_0$, and that $f_{nk}(z) = o(|z|)$ as $z \to \infty$ nontangentially.

We will require the following result.

Proposition 3.1. *Let $\Gamma_{\alpha,\beta}$ be a truncated cone, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Suppose that the arrays $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ in $\mathcal{M}$ are infinitesimal, and that the centered measures μ_{nk}° are defined as above. Then*

(1) *$E_{\mu_{nk}^{\circ}}(z) = f_{nk}(z + a_{nk})(1 + v_{nk}(z))$ for sufficiently large n, where the sequence $v_n(z) = \max_{1\le k\le k_n} |v_{nk}(z)|$ has the properties that $\lim_{n\to\infty} v_n(z) = 0$ for all $z \in \Gamma_{\alpha,\beta}$ and $v_n(z) = o(1)$ uniformly in n as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$.*

(2) *For every n, k and z, $w \in \Gamma_{\alpha,\beta}$, we have*

$$|f_{nk}(w) - f_{nk}(z)| \le |f_{nk}(z)| \frac{|z-w|}{\Im z} \left(1 + \sqrt{1+\alpha^2} \left|\frac{z}{w} - 1\right|\right).$$

(3) *For every $y > \beta$, the sequence $\{c_n + \sum_{k=1}^{k_n} E_{\mu_{nk}}(iy)\}_{n=1}^{\infty}$ converges if and only if the sequence $\{c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)]\}_{n=1}^{\infty}$ converges. Moreover, the two sequences have the same limit.*

(4) *If*

$$L = \sup_{n\ge 1} \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{t^2}{1+t^2} \, d\mu_{nk}^{\circ}(t) < +\infty,$$

then $c_n + \sum_{k=1}^{k_n} E_{\mu_{nk}}(iy) = o(y)$ uniformly in n as $y \to \infty$ if and only if $c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)] = o(y)$ uniformly in n as $y \to \infty$.

Proof. (1), (3) and (4) are proved in [21]. To prove (2), let us consider the analytic function

$$f_{\mu}(z) = \int_{-\infty}^{\infty} \frac{tz}{z-t} \, d\mu(t), \qquad z \in \mathbb{C}^+,$$

for a measure $\mu \in \mathcal{M}$. For $z, w \in \mathbb{C}^+$, we have

$$|f_{\mu}(z) - f_{\mu}(w)| \le |z-w| \int_{-\infty}^{\infty} \frac{t^2}{|w-t|\,|z-t|} \, d\mu(t)$$

and

$$\Im z \int_{-\infty}^{\infty} \frac{t^2}{|z-t|^2} \, d\mu(t) = |\Im f_{\mu}(z)| \le |f_{\mu}(z)|.$$

In addition, we have

$$\begin{aligned}\left|\frac{z-t}{w-t}\right| &\leq \frac{|z-w|+|w-t|}{|w-t|}\\ &= 1+\left|\frac{w}{w-t}\right|\left|\frac{z}{w}-1\right|\\ &\leq 1+\sqrt{1+\alpha^2}\left|\frac{z}{w}-1\right|\end{aligned}$$

for every $t \in \mathbb{R}$ and $z, w \in \Gamma_\alpha$. Therefore (2) follows from these considerations. $\square$

It was first observed in [6] that for any given truncated cone $\Gamma_{\alpha,\beta}$, the function F_μ^{-1} is defined in $\Gamma_{\alpha,\beta}$ as long as the measure μ concentrates near the origin. More precisely, for given $\alpha, \beta > 0$, there exists $\varepsilon > 0$ with the property that if $\mu \in \mathcal{M}$ is such that $\mu(\{t \in \mathbb{R} : |t| \geq \varepsilon\}) < \varepsilon$, then the function F_μ^{-1} is defined in $\Gamma_{\alpha,\beta}$.

Lemma 3.2. *Let $\Gamma_{\alpha,\beta}$ be a truncated cone, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}$. Then, for sufficiently large n, we have*

$$\Phi_{(\mu_{nk},\nu_{nk})}(z) - a_{nk} = f_{nk}(z)(1+u_{nk}(z)), \qquad z \in \Gamma_{\alpha,\beta},\ 1 \leq k \leq k_n,$$

where the sequence

$$u_n(z) = \max_{1\leq k\leq k_n} |u_{nk}(z)|$$

has the properties that $\lim_{n\to\infty} u_n(z) = 0$ for all $z \in \Gamma_{\alpha,\beta}$, and that $u_n(z) = o(1)$ uniformly in n as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$.

Proof. Introduce measures

$$d\nu_{nk}^\circ(t) = d\nu_{nk}(t + a_{nk}),$$

where the real numbers a_{nk} are defined as in (3.1). The infinitesimality of the arrays $\{\nu_{nk}\}_{n,k}$ and $\{\nu_{nk}^\circ\}_{n,k}$ and the remark we make prior to the current lemma imply, as n tends to infinity, that the functions $F_{\nu_{nk}}^{-1}$ and $F_{\nu_{nk}^\circ}^{-1}$ are defined in the cone $\Gamma_{\alpha,\beta}$ and moreover $F_{\nu_{nk}}^{-1}(z) = z(1+o(1))$ uniformly in k and $z \in \Gamma_{\alpha,\beta}$.

The desired result now follows from (1) and (2) of Proposition 3.1, and from the following observation:

$$\Phi_{(\mu_{nk},\nu_{nk})}(z) - a_{nk} = \Phi_{(\mu_{nk}^\circ,\nu_{nk}^\circ)}(z) = E_{\mu_{nk}^\circ}\left(F_{\nu_{nk}^\circ}^{-1}(z)\right) = E_{\mu_{nk}^\circ}\left(F_{\nu_{nk}}^{-1}(z) - a_{nk}\right).$$

$\square$

As shown in [8], the real and the imaginary parts of the function f_{nk} become comparable when n is large. More precisely, we have

$$|\Re f_{nk}(iy)| \leq (3+6y)\,|\Im f_{nk}(iy)|\,, \qquad 1 \leq k \leq k_n,\ y \geq 1,$$

and

$$|\Re f_{nk}(iy)| \leq 2\,|\Im f_{nk}(iy)| + |b_{nk}(y)|\,, \qquad 1 \leq k \leq k_n,\ y \geq 1,$$

where n is sufficiently large and the real-valued function $b_{nk}(y)$ is defined by

$$b_{nk}(y) = \int_{|t|\geq 1} \left[a_{nk} + \frac{(t-a_{nk})y^2}{y^2+(t-a_{nk})^2} \right] \, d\mu_{nk}(t).$$

We will need an auxiliary result from [21], where it was written in a slightly different form.

Lemma 3.3. *Consider a triangular array $\{s_{nk}\}_{n,k}$ in $[0,+\infty)$ and two arrays $\{z_{nk}\}_{n,k}$, $\{w_{nk}\}_{n,k}$ in $\mathbb{C}$. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Assume that*

(1) *$\Im w_{nk} \leq 0$ and $\Im z_{nk} \leq 0$ for all n and k;*
(2) *$z_{nk} = w_{nk}(1+\varepsilon_{nk})$ and $\lim_{n\to\infty} \varepsilon_n = 0$, where $\varepsilon_n = \max_{1\leq k\leq k_n} |\varepsilon_{nk}|$;*
(3) *there exists a constant $M > 0$ such that $|\Re w_{nk}| \leq M\,|\Im w_{nk}| + s_{nk}$ for all n and k.*

Then, for sufficiently large n, we have

$$\left| \sum_{k=1}^{k_n} [z_{nk} - w_{nk}] \right| \leq (1+M)\varepsilon_n \left| \sum_{k=1}^{k_n} \Im w_{nk} \right| + \varepsilon_n \sum_{k=1}^{k_n} s_{nk},$$

and

$$(1-\varepsilon_n - \varepsilon_n M) \left| \sum_{k=1}^{k_n} \Im w_{nk} \right| \leq \left| \sum_{k=1}^{k_n} \Im z_{nk} \right| + \varepsilon_n \sum_{k=1}^{k_n} s_{nk}.$$

In particular, if $\sup_{n\geq 1} \sum_{k=1}^{k_n} s_{nk} < +\infty$, then the sequence $\{c_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^{\infty}$ converges if and only the sequence $\{c_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ does. Moreover, the two sequences have the same limit.

Proposition 3.4. *Let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}$, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Given $\beta \geq 1$, suppose $\Gamma_{\alpha,\beta}$ is the truncated cone where the functions $\Phi_{(\mu_{nk},\nu_{nk})}$ are defined*

(1) *For every $y > \beta$, the sequence $\{c_n + \sum_{k=1}^{k_n} \Phi_{(\mu_{nk},\nu_{nk})}(iy)\}_{n=1}^{\infty}$ converges if and only if the sequence $\{c_n + \sum_{k=1}^{k_n} E_{\mu_{nk}}(iy)\}_{n=1}^{\infty}$ does. Moreover, the two sequences have the same limit.*
(2) *If $L < +\infty$ as in (4) of* Proposition 3.1, *then $c_n + \sum_{k=1}^{k_n} \Phi_{(\mu_{nk},\nu_{nk})}(iy) = o(y)$ uniformly in n as $y \to \infty$ if and only if $c_n + \sum_{k=1}^{k_n} E_{\mu_{nk}}(iy) = o(y)$ uniformly in n as $y \to \infty$.*

Proof. It was proved in [8] that $\sum_{k=1}^{k_n} |b_{nk}(y)| \leq 5yL$ for sufficiently large n and $y \geq 1$. Applying Lemmas 3.2 and 3.3 to arrays $\{f_{nk}(iy)\}_{n,k}$ and $\{\Phi_{(\mu_{nk},\nu_{nk})}(iy) - a_{nk}\}_{n,k}$, we conclude that the two sequences $\{c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(iy)]\}_{n=1}^{\infty}$ and $\{c_n + \sum_{k=1}^{k_n} \Phi_{(\mu_{nk},\nu_{nk})}(iy)\}_{n=1}^{\infty}$ have the same asymptotic behavior as $n \to \infty$. Then the proof is completed by (3) and (4) of Proposition 3.1. $\square$

We are now ready for the main result of this section. Fix real numbers γ, γ' and finite positive Borel measures σ, σ' on $\mathbb{R}$. Recall that $\nu_*^{\gamma,\sigma}$, $\nu_{\boxplus}^{\gamma,\sigma}$ and $\nu_{\uplus}^{\gamma,\sigma}$ are the $*$-, $\boxplus$-, and $\uplus$-infinitely divisible measures that we have seen in Section 2.3.

Theorem 3.5. *Let $\{c_n\}_{n=1}^\infty$ and $\{c_n'\}_{n=1}^\infty$ be two sequences in $\mathbb{R}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}$. Suppose that the sequence $\delta_{c_n'} \boxplus \nu_{n1} \boxplus \nu_{n2} \boxplus \cdots \boxplus \nu_{nk_n}$ converges weakly to $\nu_{\boxplus}^{\gamma',\sigma'}$ as $n \to \infty$. Then the following assertions are equivalent:*

(1) *The sequence $\delta_{c_n} \boxplus_c \mu_{n1} \boxplus_c \mu_{n2} \boxplus_c \cdots \boxplus_c \mu_{nk_n}$ converges weakly to $\mu \in \mathcal{M}$.*
(2) *The sequence $\delta_{c_n} \uplus \mu_{n1} \uplus \mu_{n2} \uplus \cdots \uplus \mu_{nk_n}$ converges weakly to $\nu_{\uplus}^{\gamma,\sigma}$.*
(3) *The sequence $\delta_{c_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$ converges weakly to $\nu_{\boxplus}^{\gamma,\sigma}$.*
(4) *The sequence $\delta_{c_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$ converges weakly to $\nu_*^{\gamma,\sigma}$.*
(5) *The sequence of measures*

$$d\sigma_n(t) = \sum_{k=1}^{k_n} \frac{t^2}{1+t^2}\, d\mu_{nk}^{\circ}(t)$$

converges weakly on $\mathbb{R}$ to the measure σ, and the sequence of numbers

$$\gamma_n = c_n + \sum_{k=1}^{k_n} \left[a_{nk} + \int_{-\infty}^{\infty} \frac{t}{1+t^2}\, d\mu_{nk}^{\circ}(t) \right]$$

converges to γ as $n \to \infty$.

Moreover, if (1)-(5) are satisfied, then we have $\Phi_{(\mu,\nu_{\boxplus}^{\gamma',\sigma'})} = E_{\nu_{\uplus}^{\gamma,\sigma}}$ in a truncated cone.

Proof. The equivalence of (2), (3), (4) and (5) was proved in [21]. We will show the equivalence of (1) and (2). Assume that (1) holds. Define

$$\mu_n = \delta_{c_n} \boxplus_{\mathrm{c}} \mu_{n1} \boxplus_{\mathrm{c}} \mu_{n2} \boxplus_{\mathrm{c}} \cdots \boxplus_{\mathrm{c}} \mu_{nk_n}, \ \nu_n = \delta_{c_n'} \boxplus \nu_{n1} \boxplus \nu_{n2} \boxplus \cdots \boxplus \nu_{nk_n},$$

and

$$\rho_n = \delta_{c_n} \uplus \mu_{n1} \uplus \mu_{n2} \uplus \cdots \uplus \mu_{nk_n}, \qquad n \in \mathbb{N}.$$

Then, by the weak convergence of $\{\nu_n\}_{n=1}^\infty$, there exists a truncated cone $\Gamma_{\alpha,\beta}$ such that the functions $\Phi_{(\mu_n,\nu_n)}$ are defined in $\Gamma_{\alpha,\beta}$. Thus we have

$$\Phi_{(\mu_n,\nu_n)}(z) = c_n + \sum_{k=1}^{k_n} \Phi_{(\mu_{nk},\nu_{nk})}(z)$$

in the cone $\Gamma_{\alpha,\beta}$ and

$$E_{\rho_n}(z) = c_n + \sum_{k=1}^{k_n} E_{\mu_{nk}}(z), \qquad z \in \mathbb{C}^+.$$

Also, note that

$$c_n + \sum_{k=1}^{k_n} [a_{nk} + f_{nk}(z)] = \gamma_n + \int_{-\infty}^{\infty} \frac{1+tz}{z-t}\, d\sigma_n(t), \tag{3.2}$$

and that the quantity L as in (4) of Proposition 3.1 is precisely $\sup_{n\geq 1}\sigma_n(\mathbb{R})$.

Propositions 2.4, 3.1 and 3.4 imply that

$$\lim_{n\to\infty} E_{\rho_n}(iy) = \Phi_{(\mu,\nu_{\boxplus}^{\gamma',\sigma'})}(iy) = \lim_{n\to\infty}\left(c_n + \sum_{k=1}^{k_n}[a_{nk}+f_{nk}(iy)]\right), \qquad y>\beta.$$

Since $\{c_n + \sum_{k=1}^{k_n}[a_{nk}+f_{nk}]\}_{n=1}^{\infty}$ is a normal family, an application of Montel's theorem shows that the sequence $\{c_n+\sum_{k=1}^{k_n}[a_{nk}+f_{nk}(i)]\}_{n=1}^{\infty}$ converges to $\Phi_{(\mu,\nu_{\boxplus}^{\gamma',\sigma'})}(i)$. Hence (3.2) implies that

$$\begin{aligned}\lim_{n\to\infty}\sigma_n(\mathbb{R}) &= \lim_{n\to\infty} -\Im\left(c_n+\sum_{k=1}^{k_n}[a_{nk}+f_{nk}(i)]\right)\\ &= -\Im\Phi_{(\mu,\nu_{\boxplus}^{\gamma',\sigma'})}(i) < +\infty.\end{aligned}$$

We deduce that $L=\sup_{n\geq 1}\sigma_n(\mathbb{R})<+\infty$, and therefore (2) holds by Propositions 2.4 and 3.4. Moreover, in this case we have $\Phi_{(\mu,\nu_{\boxplus}^{\gamma',\sigma'})} = E_{\nu_{\uplus}^{\gamma,\sigma}}$ in the cone $\Gamma_{\alpha,\beta}$ by the uniqueness principle in complex analysis.

Conversely, suppose now (2) holds. Using the equivalence of (2) and (5), we see that $L<+\infty$ and hence (1) follows again from Propositions 2.4 and 3.4. $\square$

Theorem 3.5 shows that the reciprocal of the Cauchy transform of the limit law μ is given by

$$F_\mu(z) = z - E_{\nu_{\uplus}^{\gamma,\sigma}}\left(F_{\nu_{\boxplus}^{\gamma',\sigma'}}(z)\right), \qquad z\in\mathbb{C}^+. \tag{3.3}$$

Therefore, in order to determine the limit law μ, one first finds the parameters γ, γ', σ and σ' by (5) of Theorem 3.5, then uses the formulas (2.4) and (2.5) to obtain the function F_μ from (3.3). Finally, the measure μ is recovered from the function G_μ as we have seen in Section 2.1.

In this spirit, we see that the results in [9] concerning the c-free analogues of the central and Poisson limit theorems are direct consequences of Theorem 3.5. Indeed, given $\alpha,\beta\geq 0$, in case $\gamma=\gamma'=0$, $\sigma=\alpha^2\delta_0$ and $\sigma'=\beta^2\delta_0$, the limit law μ is a c-free version of the centered Gaussian distribution on $\mathbb{R}$ which appeared in [9, Theorem 4.3]. A c-free analogue of the Poisson law as in [9, Theorem 4.4] is obtained when $\gamma=\alpha/2$, $\gamma'=\beta/2$, $\sigma=(\alpha/2)\delta_1$ and $\sigma'=(\beta/2)\delta_1$.

It is also interesting to note that (3.3) shows that the limit law $\mu=\delta_0$ if and only if $\gamma=0$ and the measure $\sigma=\delta_0$. Thus, by Theorem 3.5, one obtains necessary and sufficient conditions for the weak convergence to the Dirac measure at the origin, which can be viewed as the c-free analogue of the weak law of large numbers.

4. Application to the $\boxplus_c$-infinite Divisibility

In this section we give various characterizations of the $\boxplus_c$-infinite divisibility with the help of Theorem 3.5. The analogue of Theorem 4.1 for compactly supported

measures was obtained earlier in [15] by analyzing the solutions of a complex Burger's equation. The approach we presented here deals with general probability measures, and does not involve such a differential equation.

Before outlining the main result we need a definition. A family of pairs $\{(\mu_t,\nu_t)\}_{t\geq 0}$ of probability measures on $\mathbb{R}$ is said to be a *weakly continuous semigroup* relative to the convolution $\boxplus_{\mathrm{c}}$ if $(\mu_t,\nu_t)\boxplus_{\mathrm{c}}(\mu_s,\nu_s)=(\mu_{t+s},\nu_{t+s})$ for $t,s\geq 0$, and the maps $t\mapsto\mu_t$ and $t\mapsto\nu_t$ are continuous.

Theorem 4.1. *Given a $\boxplus$-infinitely divisible measure $\nu\in\mathcal{M}$ and a measure $\mu\in\mathcal{M}$, the following statements are equivalent:*

1. *The pair (μ,ν) is $\boxplus_c$-infinitely divisible.*
2. *There exists a real number γ and a finite positive Borel measure σ on $\mathbb{R}$ such that the function*
$$\Phi_{(\mu,\nu)}(z)=\gamma+\int_{-\infty}^{\infty}\frac{1+tz}{z-t}\,d\sigma(t),\qquad z\in\mathbb{C}^+.$$
3. *The function $\Phi_{(\mu,\nu)}$ can be analytically continued to $\mathbb{C}^+$.*
4. *There exists a weakly continuous semigroup $\{(\mu_t,\nu_t)\}_{t\geq 0}$ relative to $\boxplus_c$ such that $(\mu_0,\nu_0)=(\delta_0,\delta_0)$ and $(\mu_1,\nu_1)=(\mu,\nu)$.*

Moreover, if statements (1) *to* (4) *are all satisfied, then the limit*
$$\gamma=\lim_{t\to 0^+}\left[\frac{1}{t}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,d\mu_t(x)\right]$$
exists and the measure σ is the weak limit of measures
$$\frac{1}{t}\,\frac{x^2}{1+x^2}\,d\mu_t(x)$$
as $t\to 0^+$.

Proof. We first prove that (1) implies (2). Assume that (1) holds. For every $n\in\mathbb{N}$, we have
$$\mu=\underbrace{\mu_n\boxplus_{\mathrm{c}}\mu_n\boxplus_{\mathrm{c}}\cdots\boxplus_{\mathrm{c}}\mu_n}_{n\,\text{times}}\qquad\text{and}\qquad\nu=\underbrace{\nu_n\boxplus\nu_n\boxplus\cdots\boxplus\nu_n}_{n\,\text{times}},$$
where $\mu_n,\nu_n\in\mathcal{M}$. Then we have $F_{\nu_n}^{-1}(z)-z=\left[F_\nu^{-1}(z)-z\right]/n$, and hence the measures ν_n converge weakly to δ_0 as $n\to\infty$ by Proposition 2.3 in [5]. On the other hand, the identity $\Phi_{(\mu_n,\nu_n)}(z)=\Phi_{(\mu,\nu)}(z)/n$ and Proposition 2.4 imply that the measures μ_n converge weakly to δ_0 as well. Let us introduce two infinitesimal arrays $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ by setting $\mu_{nk}=\mu_n$ and $\nu_{nk}=\nu_n$, where $1\leq k\leq n$. Then the measure μ (resp., ν) can be viewed as the weak limit of the c-free (resp., free) convolutions $\mu_{n1}\boxplus_{\mathrm{c}}\mu_{n2}\boxplus_{\mathrm{c}}\cdots\boxplus_{\mathrm{c}}\mu_{nn}$ (resp., $\nu_{n1}\boxplus\nu_{n2}\boxplus\cdots\boxplus\nu_{nn}$). Hence (2) follows form Theorem 3.5.

The equivalence of (2) and (3) is based on the Nevanlinna integral representation of analytic self-mappings in $\mathbb{C}^+$(see [1]).

We next show that (2) implies (4). Suppose that (2) holds. It was proved in [7] that there exists a weakly continuous semigroup $\{\nu_t\}_{t\geq 0}$ relative to $\boxplus$ so that $\nu_0 = \delta_0$ and $\nu_1 = \nu$. Then, for every $t \geq 0$, there exists a unique probability measure μ_t on $\mathbb{R}$ such that $E_{\mu_t}(z) = t\left(\Phi_{(\mu,\nu)}\left(F_{\nu_t}(z)\right)\right)$ for all $z \in \mathbb{C}^+$, where $\mu_0 = \delta_0$. It is easy to see that the c-free convolution semigoup $\{(\mu_t, \nu_t)\}_{t\geq 0}$ has the desired properties.

The implication form (4) to (1) is obvious. To finish the proof, we only need to show the assertions about the measure σ and the number γ. Assume that the pair (μ, ν) is $\boxplus_{\rm c}$-infinitely divisible, and let $\{(\mu_t, \nu_t)\}_{t\geq 0}$ be the corresponding convolution semigroup as in (4). Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} t_n = 0$. Let $k_n = [1/t_n]$ for every $n \in \mathbb{N}$, where $[x]$ denotes the largest integer that is no greater than the real number x. Observe that

$$1 - t_n < t_n k_n \leq 1, \qquad n \in \mathbb{N}.$$

Hence we have $\lim_{n\to\infty} t_n k_n = 1$, and further the properties of the semigroup $\{(\mu_t, \nu_t)\}_{t\geq 0}$ show that the c-free convolutions

$$\underbrace{\mu_{t_n} \boxplus_{\rm c} \mu_{t_n} \boxplus_{\rm c} \cdots \boxplus_{\rm c} \mu_{t_n}}_{k_n \text{ times}} = \mu_{t_n k_n}$$

converge weakly to the measure $\mu_1 = \mu$ as $n \to \infty$. Theorem 3.5 then implies that the measures

$$\frac{1}{t_n}\,\frac{x^2}{1+x^2}\,d\mu_{t_n}^{\circ}(x) = \frac{1}{t_n k_n}\,k_n \frac{x^2}{1+x^2}\,d\mu_{t_n}^{\circ}(x)$$

converge weakly to the measure σ and

$$\gamma = \lim_{n\to\infty}\left[\frac{1}{t_n}\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,d\mu_t^{\circ}(x)\right],$$

where the centered measures $d\mu_{t_n}^{\circ}(x) = d\mu_{t_n}(x + a_n)$ and the numbers a_n are defined as in (3.1). The desired result follows from the facts that $\lim_{n\to\infty} a_n = 0$, and that the topology on the set $\mathcal{M}$ determined by the weak convergence of measures is actually metrizable [12, Problem 14.5]. ☐

We conclude this section by showing a result, which is a c-free analogue of Hinčin's classical theorem on the $*$-infinite divisibility [14].

Corollary 4.2. *Let $\{c_n\}_{n=1}^{\infty}$ and $\{c'_n\}_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}$. Suppose that the sequence $\delta_{c_n} \boxplus_c \mu_{n1} \boxplus_c \mu_{n2} \boxplus_c \cdots \boxplus_c \mu_{nk_n}$ converges weakly to μ, and that the sequence $\delta_{c'_n} \boxplus \nu_{n1} \boxplus \nu_{n2} \boxplus \cdots \boxplus \nu_{nk_n}$ converges weakly to ν. Then the pair (μ, ν) is $\boxplus_c$-infinitely divisible.*

Proof. It was proved in [6] that the measure ν must be $\boxplus$-infinitely divisible. Therefore the result follows immediately from Theorems 3.5 and 4.1. ☐

In this section we determine all $\boxplus_c$-stable pairs of measures, which are defined as follows. Denote by $\mathcal{M} \times \mathcal{M}$ the set of all pairs of measures (μ, ν), where $\mu, \nu \in \mathcal{M}$. Two pairs of measures (μ_1, ν_1) and (μ_2, ν_2) in $\mathcal{M} \times \mathcal{M}$ are said to be *equivalent* if there exist real numbers a, b, with $a > 0$, such that $d\mu_2(t) = d\mu_1(at + b)$ and $d\nu_2(t) = d\nu_1(at + b)$; we indicate this by writing $(\mu_1, \nu_1) \sim (\mu_2, \nu_2)$. By analogy with classical probability theory, we say a pair of measures $(\mu, \nu) \in \mathcal{M} \times \mathcal{M}$ is *$\boxplus_c$-stable* if $(\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) \sim (\mu, \nu)$ whenever $(\mu_1, \nu_1) \sim (\mu, \nu) \sim (\mu_2, \nu_2)$.

Remark 5.1. Note that if $d\mu_2(t) = d\mu_1(at+b)$ and $d\nu_2(t) = d\nu_1(at+b)$, where $a > 0$, then (2.2) shows that

$$\Phi_{(\mu_2,\nu_2)}(z) = \frac{1}{a}\left[\Phi_{(\mu_1,\nu_1)}(az) - b\right] \tag{5.1}$$

in a truncated cone. Conversely, if pairs (μ_1, ν_1) and (μ_2, ν_2) are such that $d\nu_2(t) = d\nu_1(at + b)$, where $a > 0$, and (5.1) holds in a truncated cone, then

$$d\mu_2(t) = d\mu_1(at + b).$$

Proposition 5.2. *If (μ, ν) is $\boxplus_c$-stable, then (μ, ν) is $\boxplus_c$-infinitely divisible.*

Proof. The $\boxplus_c$-stability of (μ, ν) implies that $(\mu \boxplus_c \mu, \nu \boxplus \nu) = (\mu, \nu) \boxplus_c (\mu, \nu) \sim (\mu, \nu)$, that is, there exist $a_2 > 0$ and $b_2 \in \mathbb{R}$ such that

$$d\mu(t) = d(\mu \boxplus_c \mu)(a_2 t + b_2) \qquad \text{and} \qquad d\nu(t) = d(\nu \boxplus \nu)(a_2 t + b_2).$$

The analytic description of free convolution implies that

$$\begin{aligned} F_\nu^{-1}(z) &= \frac{1}{a_2}\left[F_{\nu\boxplus\nu}^{-1}(a_2 z) - b_2\right] \\ &= \frac{2}{a_2}\left[F_\nu^{-1}(a_2 z) - \frac{b_2}{2}\right] - z \\ &= 2F_{\nu_2}^{-1}(z) - z = F_{\nu_2\boxplus\nu_2}^{-1}(z), \end{aligned}$$

where $d\nu_2(t) = d\nu(a_2 t + b_2/2)$. This shows that $\nu = \nu_2 \boxplus \nu_2$. Moreover, Remark 5.1 and Proposition 2.2 show that

$$\begin{aligned} \Phi_{(\mu,\nu)}(z) &= \frac{1}{a_2}\left[\Phi_{(\mu\boxplus_c\mu,\nu\boxplus\nu)}(a_2 z) - b_2\right] \\ &= \frac{2}{a_2}\left[\Phi_{(\mu,\nu)}(a_2 z) - \frac{b_2}{2}\right] \\ &= 2\Phi_{(\mu_2,\nu_2)}(z) = \Phi_{(\mu_2\boxplus_c\mu_2,\nu_2\boxplus\nu_2)}(z) \end{aligned}$$

in a truncated cone, where $d\mu_2(t) = d\mu(a_2 t + b_2/2)$. Therefore, we have $\mu = \mu_2 \boxplus_c \mu_2$.

Next, we consider $(\mu_2, \nu_2) \sim (\mu, \nu) = (\mu_2 \boxplus_c \mu_2, \nu_2 \boxplus \nu_2)$. By a slight modification of the above argument, it is easy to verify that there exist $a_3 > 0$ and $b_3 \in \mathbb{R}$ such that $\nu = \nu_3 \boxplus \nu_3 \boxplus \nu_3$ and $\mu = \mu_3 \boxplus_c \mu_3 \boxplus_c \mu_3$, where $d\nu_3(t) = d\nu_2(a_3 t + b_3/3)$ and

$d\mu_3(t) = d\mu_2(a_3 t + b_3/3)$. Continuing in this fashion, we see that the pair (μ, ν) is $\boxplus_c$-infinitely divisible. $\square$

Recall from [7] that an analytic function $\phi : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ is said to be *stable* if for every $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}$ such that

$$\phi(z) + \frac{1}{a}\phi(az) = \frac{1}{b}\phi(bz) + c, \qquad z \in \mathbb{C}^+.$$

The next result follows immediately from Remark 5.1.

Proposition 5.3. *A $\boxplus_c$-infinitely divisible pair of measures (μ, ν) is $\boxplus_c$-stable if and only if the functions $\Phi_{(\mu,\nu)}$ and $F_\nu^{-1}(z) - z$ are stable.*

A complete characterization of stable analytic functions was proved in [7]. We will write out this result below for the sake of completeness. The complex functions in the following list are given by their principal value in the upper half plane.

Theorem 5.4. *The following is a complete list of the stable analytic functions $\phi : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$.*

(1) $\phi(z) = a + ib$, $a \in \mathbb{R}$ *and* $b \leq 0$.
(2) $\phi(z) = a + bz^{-\alpha+1}$, $a \in \mathbb{R}$, $\alpha \in (1, 2]$, $b \neq 0$, *and* $\arg b \in [(\alpha - 2)\pi, 0]$.
(3) $\phi(z) = a + bz^{-\alpha+1}$, $a \in \mathbb{R}$, $\alpha \in (0, 1)$, $b \neq 0$, *and* $\arg b \in [-\pi, (\alpha - 1)\pi]$.
(4) $\phi(z) = a + b \log z$, $a \in \mathbb{C}^- \cup \mathbb{R}$ *and* $b < 0$.

Finally, we briefly outline the role of $\boxplus_c$-stable pairs of measures in relation to the limit theorems. Following the ideas in [16], one can show that a pair of measures (μ, ν) is $\boxplus_c$-stable if and only if there exist $A_n > 0$, $B_n \in \mathbb{R}$ and measures $\mu', \nu' \in \mathcal{M}$ so that the measure μ (resp., ν) is the weak limit of c-free (resp., free) convolutions $\underbrace{\mu_n \boxplus_c \mu_n \boxplus_c \cdots \boxplus_c \mu_n}_{n\,\text{times}}$ (resp., $\underbrace{\nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n}_{n\,\text{times}}$), where the measure μ_n and ν_n are given by

$$d\mu_n(t) = d\mu'(A_n t + B_n), \qquad \text{and} \qquad d\nu_n(t) = d\nu'(A_n t + B_n).$$

We will not provide the details of the proof of the above assertion because it is quite similar to those in the free case [16]. The reader will have no difficulty in providing his/her own proof.

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